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Journal of Sound and Vibration 271 (2004) 789–813
JOURNAL OF SOUND AND VIBRATION
www.elsevier.com/locate/jsvi

An amplitude equation for the non-linear vibration of viscoelastically damped sandwich beams

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Received 23 August 2002; accepted 3 March 2003

Abstract

An elementary theory for non-linear vibrations of viscoelastic sandwich beams is presented. The harmonic balance method is coupled with a one mode Galerkin analysis. This results in a scalar complex frequency–response relationship. So the non-linear free vibration response is governed by only two complex numbers. This permits one to recover first the concept of linear loss factor, second a parabolic approximation of the backbone curve that accounts for the amplitude dependence of the frequency. A new amplitude–loss factor relationship is also established in this way. The forced vibration analysis leads to resonance curves that are classical within non-linear vibration theory. They are extended here to any viscoelastic constitutive behaviour.

This elementary approach could be extended to a large class of structures and in a finite element framework. The amplitude equation is obtained in closed form for a class of sandwich beams. The effects of the boundary conditions and of the temperature on the response are discussed.

1. Introduction

Typical viscoelastically damped structures are of sandwich construction in which a thin viscoelastic layer is sandwiched between identical elastic layers. These structures are used in many areas (e.g., aerospace industry) for vibration and noise control thanks to their superior capability in energy absorption. They particularly offer the advantage of high damping with low weight addition. The interlayer damping concept is highly compatible with the laminated configuration

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0022-460X/$ - see front matter
doi:10.1016/S0022-460X(03)00754-5

of composite structures and with their fabrication techniques and provides an effective way to reduce vibrations and noise in structures. The damping is introduced by an important transverse shear in the viscoelastic layer. It is due to the difference between in-plane displacements of the elastic layers and also to the low stiffness of the central layer.

Many investigations have been devoted to the linear dynamic analysis of these structures. Their stiffness matrix is a complex one and it depends non-linearly on the vibration frequency. So, the linear vibration analysis leads to a non-linear complex eigenproblem. Solving this problem yields firstly, complex modes that can be slightly different from undamped modes and secondly, complex eigenvalues whose real and imaginary parts are related to the loss factors and to the viscous eigenfrequencies. From an engineering point of view, the most relevant quantity is the loss factor that is associated with any mode. Several methods have been presented to predict these structural damping properties: some analytical studies [1–9] were devoted to simple structures, and finite element simulations [10–21] were introduced to design structures with complex geometries and generic boundary conditions. The simplest technique is the modal strain energy method [11] that defines a rather good estimate of the loss factor from a sort of one mode Galerkin approximation. In most of these studies the viscoelastic behaviour is simply accounted for by a complex modulus, but nowadays one is able to compute the loss factor with any linear viscoelastic constitutive law [19].

It must be stressed, however, that non-linear systems can show a behaviour that is qualitatively different from that of linear systems and the behaviour for large amplitudes can differ significantly from the behaviour for small amplitudes [22,23]. The frequency of non-linear elastic systems depends on the vibration amplitude. In a first approximation, the so-called backbone curve is parabolic as follows:

$$\left(\frac{\omega}{\omega_L}\right)^2 = 1 + C\left(\frac{a}{h}\right)^2, \tag{1}$$

where ω is the non-linear frequency, a is the vibration amplitude, ω_L is the linear frequency and h is a typical length such as plate thickness. Thus, the non-linear dynamic response of a structure can be represented by a single non-dimensional number C that is a sort of non-linear dynamic stiffness. The harmonic balance method is an elementary way to characterize this non-linear response as well as the evolution of the mode itself [24–26]. Nevertheless, this technique is too intricate in view of a simple engineering analysis. In this respect, the harmonic balance method has to be coupled with a single mode Galerkin approximation. The coupling of the two approximations (harmonic balance, one spatial mode) leads to representation (1) of the backbone curve. One may refer for instance to Refs. [27,28] for a discussion of the validity of the latter approximation.

However, there are only few studies about the non-linear analysis of damped vibrations of sandwich structures. Kovac et al. [29] and Hyer et al. [30,31] studied the non-linear vibrations of damped sandwich beams. The non-linear damped free vibrations of sandwich plates and cylindrical panels have been investigated by Xia and Lukasiewicz [32,33]. These analyses are based on multimode Galerkin's procedure and harmonic balance method; the main limitation lies in the viscoelastic model that is of the Kelvin–Voigt type. More recently, Ganapathi et al. [34] introduced the concept of a loss factor that depends on the amplitude. They considered flexural–torsional motions of beams and plates and their analysis is similar to that of many recent papers about linear damping of structures: finite element framework, damping accounted by a complex modulus, complex eigenproblems solved by the QR method.

The goal of this paper is to establish the simplest consistent theory for the non-linear vibration analysis of a viscoelastic sandwich beam. The non-linearity arises from axial stretching of the elastic face layers and the damping from the shear deformation of the viscoelastic core. The starting sandwich model and the mathematical treatment are chosen in order to provide the simplest consistent theory. Thus, the present study is limited to periodic or damped responses and couples the harmonic balance method with a one mode Galerkin approximation. The result is an amplitude equation that is similar to a classical bifurcation equation:

$$-\omega^2 MA + KA + K_{nl}A|A|^2 = Q, \tag{2}$$

where A is the unknown complex amplitude and Q is the modal force. Then, for each mode, the free vibrations depend on three constants M, K, K_{nl}. M is a sort of modal mass. The complex constants K and K_{nl} are, respectively, a linear and a non-linear stiffness. This implies that the free vibrations are governed by two complex constants K/M and K_{nl}/M. The physical meaning of these constants, which can be connected with the aforementioned concepts: loss factor, backbone constant C appearing in Eq. (1) and non-linear damping will be discussed. In the forced vibration case, Eq. (2) permits one to recover classical resonance curves, that are deduced here for any viscoelastic stress–strain law.

2. A non-linear model for three-layered viscoelastic beam

The simplest possible model that is able to account correctly for the effect of geometrical non-linearities on the response of an elastic/viscoelastic/elastic sandwich beam is presented. The basic assumptions are consistent with the classical linear sandwich analysis [5]: the two elastic layers remain parallel and there is a shear strain that does not vary across the thickness of the viscoelastic core. A unique laminate model based on Bernoulli or Timoshenko kinematical assumptions would be simpler, but it would underestimate the shear strain in a soft core. Sometimes more refined descriptions of the stress field in the core are considered for aeronautical structures (see for instance Ref. [34]), but this is not necessary in the case of a thin and soft core. The geometrical non-linearity is represented according to the classical framework of small strain and finite deflection [25].

2.1. Kinematics of the model

Consider a three-layer symmetric sandwich beam with a viscoelastic core as pictured in Fig. 1. Let x be the mid-surface co-ordinate of the undeformed beam and z is the one transverse to the thickness. Let z_i be the co-ordinate of the middle of the layer i. Because of symmetry, $z_1 = (h_f + h_c)/2 = -z_3,\ z_2 = 0$. h_f is the thickness of the elastic layer and h_c denotes the thickness of the core. In order to evaluate accurately the damping in the sandwich structures, one has to take into account the shear deformation in the viscoelastic layer. This shear results from the difference between the in-plane displacements of the elastic layers. So, the following hypotheses,

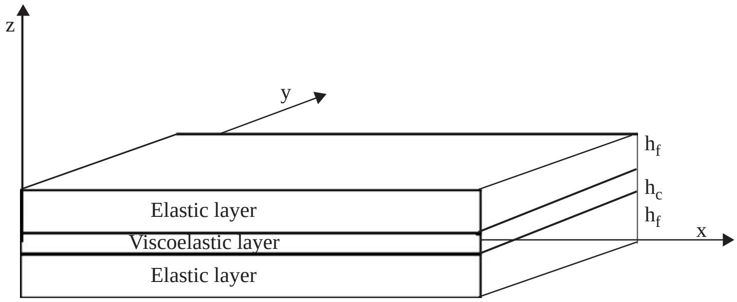

Fig. 1. Sandwich structure with viscoelastic middle layer.

common to many authors [5–9], are assumed:

- All points on a normal to the beam axis have the same transverse displacement $w(x,t)$.
- The displacement is continuous along the interfaces between central and elastic layers.
- All points of the elastic layers on a normal have the same rotations.
- The core material is homogeneous, isotropic and viscoelastic. So, the Young's modulus is complex and depends on the vibration frequency. As in most analyses, the Poisson ratio υ_c is assumed to be constant.
- The elastic layers have the same Young modulus E_f.

Assuming negligible shear deformation in the face layers, the longitudinal and transverse displacement U_i and W_i at any point within the face layers may be written as

$$\begin{aligned} U_i(x,z,t) &= u_i(x,t) - (z - z_i)\frac{\partial w}{\partial x} \quad i = 1,3, \\ W_i(x,z,t) &= w(x,t) \end{aligned} \tag{3}$$

where $u_i(x,t)$ is the axial displacement of the middle of the ith layer and $w(x,t)$ is the common transverse displacement. As in Ref. [8], the displacement of the central layer is written in the following form:

$$\begin{aligned} U_2(x,z,t) &= u(x,t) + z\beta(x,t), \\ W_2(x,z,t) &= w(x,t) \end{aligned} \tag{4}$$

where $u(x,t)$ is the axial displacement of the centreline of the core and β is the rotation of the normal to the mid-plane in the viscoelastic core. Remember that the transverse displacement is assumed to be the same for any layer. According to the assumption of small strains and moderate rotations, the non-linear strain–displacement relations for each layer are assumed in the following form [25]:

$$\begin{aligned} \varepsilon_i(x,z,t) &= \frac{\partial u_i}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 - (z - z_i)\frac{\partial^2 w}{\partial x^2}, \quad i = 1,3, \\ \varepsilon_2(x,z,t) &= \varepsilon(x,t) + z\frac{\partial \beta}{\partial x}, \quad \varepsilon(x,t) = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2. \end{aligned} \tag{5}$$

2.2. Constitutive behaviour

According to Hooke stress–strain relation, the axial forces and bending moments in the elastic layers are expressed as

$$N_1 = E_f S_f \left(\frac{\partial u_1}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right),$$
$$N_3 = E_f S_f \left(\frac{\partial u_3}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \right),$$
$$M_1 = M_3 = E_f I_f \frac{\partial^2 w}{\partial x^2}, \tag{6}$$

where S_f is the cross-section area and I_f is the second moment of the faces.

A general linear viscoelastic constitutive law is assumed for the core. Classically, such a law involves convolution products [30,31,36]. For instance, the relation between the axial strain and stress can be written in the form of a convolution product:

$$\sigma_2(x,t) = \int_{-\infty}^{t} Y(t-\tau) \frac{\partial}{\partial \tau} \varepsilon(x,\tau)\, d\tau = Y * \frac{\partial \varepsilon}{\partial t}, \tag{7}$$

where Y is the relaxation function of the viscoelastic material. In the following, the constitutive viscoelastic law will be used when the strain is a combination of various harmonics ω_j as follows:

$$\varepsilon(x,t) = \sum_{j=1}^{k} \{\varepsilon(x,j) e^{i\omega_j t} + CC\}, \tag{8}$$

where CC denotes the conjugate complex. In the next part, splitting (8) will follow from the assumption of a harmonic deflection and not from a superposition principle as in a linear framework. The beam equations can be deduced from Eqs. (7), (8) and by introducing the normal force in the core N_2. After some manipulations, one gets

$$N_2(x,t) = \sum_{j=1}^{k} \{N_2(x,j) e^{i\omega_j t} + CC\},$$
$$N_2(x,j) = S_c E_c(\omega_j) \varepsilon(x,j),$$
$$E_c(\omega_j) = i\omega_j \int_0^{+\infty} Y(\xi) e^{-i\omega_j \xi}\, d\xi, \tag{9}$$

where E_c is the complex Young modulus and S_c is the cross-section area of the core. Constitutive laws for the bending moment M_2 and the shear force T of the core are obtained using the same procedure. Note that a Timoshenko shear coefficient is not necessary in such sandwich structures [5].

$$M_2 = I_c Y * \left\{ \frac{\partial}{\partial x} \frac{\partial \beta}{\partial t} \right\},$$
$$T = \frac{S_c}{2(1+\upsilon_c)} Y * \left\{ \frac{\partial}{\partial x} \frac{\partial w}{\partial t} + \frac{\partial \beta}{\partial t} \right\}. \quad (10)$$

2.3. Balance of momentum

Bernoulli and Timoshenko kinematical assumptions have been considered for the faces and for the core, respectively. Hence, the virtual work principle for a geometrically non-linear sandwich can be written by assembling the virtual work equations of the three layers, each one being a non-linear beam. This leads to

$$\sum_{i=1}^{3} \int_0^L N_i \left[\frac{\partial \delta u_i}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial \delta w}{\partial x} \right] \mathrm{d}x + \int_0^L \left[(M_1 + M_3) \frac{\partial^2 \delta w}{\partial x^2} + M_2 \frac{\partial \delta \beta}{\partial x} \right] \mathrm{d}x$$
$$+ \int_0^L T \left[\frac{\partial \delta w}{\partial x} + \delta \beta \right] \mathrm{d}x = \delta P_{ext} - \delta P_{acc}, \quad (11)$$

where P_{ext} is the potential energy of the external load and δP_{acc} is the virtual work of inertial terms. It is customary to neglect the influence of axial inertia effects when dealing with flexural response of beams [24,28]. Thus, expressions of P_{ext} and P_{acc} are

$$\delta P_{ext} = \int_0^L F(x,t) \delta w(x,t) \, \mathrm{d}x,$$
$$\delta P_{acc} = (2\rho_f S_f + \rho_c S_c) \int_0^L \frac{\partial^2 w}{\partial t^2} \delta w(x,t) \, \mathrm{d}x \quad (12)$$

in which $F(x,t)$ is the external load and ρ_f and ρ_c are the mass density of faces and core, respectively. In the present work, the load is assumed to be harmonic in time, transversal and does not depend on the displacement.

Considering the continuity conditions of the displacements at the interfaces between the central and the face layers, one can express the face displacements u_1 and u_3 as follows:

$$u_1(x,t) = u(x,t) + \left[\frac{h_c}{2} \beta(x,t) - \frac{h_f}{2} \frac{\partial w(x,t)}{\partial x} \right],$$
$$u_3(x,t) = u(x,t) - \left[\frac{h_c}{2} \beta(x,t) - \frac{h_f}{2} \frac{\partial w(x,t)}{\partial x} \right]. \quad (13)$$

Based on these interface conditions, it is possible to eliminate all the core variables from the potential and kinetic energy expressions. So all the variables would be written in terms of face displacements only. This procedure is applied by many authors [5,7,8]. In the present analysis, the core variables will be used and the number of independent generalized displacements is reduced to three, namely u, w and β.

In this model, the internal forces are, on one hand, the shear stress T that is due to the shear deformation of the core, on the other hand the following three quantities:

$$N = N_1 + N_2 + N_3, \quad M_\beta = M_2 + \frac{N_1 - N_3}{2} h_c,$$
$$M_w = M_1 + M_3 - \frac{N_1 - N_3}{2} h_f. \tag{14}$$

According to identity (13) and to definitions (14), the variational formulation (11) can be rewritten as

$$\int_0^L \left\{ N\left(\frac{\partial \delta u}{\partial x} + \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x}\right) + M_\beta \frac{\partial \delta\beta}{\partial x} + M_w \frac{\partial^2 \delta w}{\partial x^2} + T\left(\frac{\partial \delta w}{\partial x} + \delta\beta\right)\right\} \mathrm{d}x$$
$$= \delta P_{ext} - \delta P_{acc}. \tag{15}$$

Thus, the stress state in the sandwich is described by four scalar quantities: the total axial force N, the shear force T and two bending moments M_β and M_w. The global moment M_β–M_w is recovered only if the virtual displacement satisfies the Bernoulli condition $\delta\beta = -\partial\delta w/\partial x$.

Because the axial inertial term and the axial excitation force are disregarded, variational equation (15) implies that the normal force N does not depend on x. Thus the balance of momentum (15) is split in two equations as follows:

$$N(x,t) = 2E_f S_f \varepsilon + S_c E_c * \frac{\partial \varepsilon}{\partial t} = N_0(t),$$
$$\varepsilon = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \tag{16a}$$

$$\int_0^L \left\{ N \frac{\partial w}{\partial x}\frac{\partial \delta w}{\partial x} + M_\beta \frac{\partial \delta\beta}{\partial x} + M_w \frac{\partial^2 \delta w}{\partial x^2} + T\left(\frac{\partial \delta w}{\partial x} + \delta\beta\right)\right\} \mathrm{d}x$$
$$= \int_0^L F(x,t)\delta w \,\mathrm{d}x - (2\rho_f S_f + \rho_c S_c)\int_0^L \frac{\partial^2 w}{\partial t^2}\delta w \,\mathrm{d}x,$$
$$M_\beta = I_c Y * \left\{\frac{\partial}{\partial x}\frac{\partial \beta}{\partial t}\right\} + \frac{E_f S_f h_c}{2}\left(h_c \frac{\partial \beta}{\partial x} - h_f \frac{\partial^2 w}{\partial x^2}\right),$$
$$M_w = E_f\left(2I_f + \frac{S_f h_f^2}{2}\right)\frac{\partial^2 w}{\partial x^2} - \frac{E_f S_f h_c h_f}{2}\frac{\partial \beta}{\partial x},$$
$$T(x,t) = \frac{S_c}{2(1+\upsilon_c)} Y * \left\{\frac{\partial \beta}{\partial t} + \frac{\partial}{\partial x}\frac{\partial w}{\partial t}\right\}. \tag{16b}$$

In summary, one has to find exact or approximate solutions of an integro-differential system, which include Eqs. (16), (6) and (10). As compared with the few papers about the non-linear behaviour of viscoelastic sandwich structures [29–35], the key point is the account of a general viscoelastic modelling for the core.

3. Harmonic approximate solutions

In this paper, the analysis is limited to periodic responses to a transverse harmonic excitation $F(x,t) = F(x)\cos(\omega t)$ or to free vibrations. With a view to obtain the simplest approximate analysis, the dependence in time and in space has to be specified in a very restrictive way. So the deflection will be assumed to be harmonic in time, parallel to a single mode in space and its complex amplitude being arbitrary. Eqs. (16a) for the axial quantities $u(x,t)$ and $N(x,t)$ will be solved exactly: this is possible for straight members or flat plates within the von Karman framework, if the axial inertia term is neglected. The bending equations (16b) will be reduced to a single complex equation, by using the harmonic balance method and a one-mode Galerkin approximation in space. This will lead to a non-linear frequency–amplitude relationship, that will govern the non-linear viscoelastic response. Here, the analysis begins with the harmonic balance reduction to take advantage of the simplicity of the viscoelastic law for harmonic motions. Note that this approach differs slightly from the current practice for non-linear structural vibrations, that was used in most of previous studies [29–33]: generally a multimode Galerkin approximation is first performed, that is sometimes followed by a harmonic balance reduction to characterize the periodic solutions.

3.1. Linear vibration mode and modal approximation

The linear vibration mode corresponds to the eigenmode $\{W(x), B(x)\}$ of the undamped sandwich beam. This eigenmode and the corresponding eigenfrequency ω_0 satisfy the following eigenvalue problem:

$$\int_0^L [M_\beta \delta B' + M_w \delta W'' + T(\delta W' + \delta B)]\,\mathrm{d}x = (2\rho_f S_f + \rho_c S_c)\omega_0^2 \int_0^L W \delta W \mathrm{d}x, \tag{17}$$

where

$$\begin{aligned} M_\beta &= \left(E_c(0)I_c + \frac{E_f S_f h_c^2}{2}\right)B' - \frac{E_f S_f h_c h_f}{2} W'', \\ M_w &= E_f\left(2I_f + \frac{S_f h_f^2}{2}\right)W'' - \frac{E_f S_f h_c h_f}{2} B', \\ T &= G_c(0)S_c(W' + B), \\ G_c(0) &= \frac{E_c(0)}{2(1+v_c)} \end{aligned} \tag{18}$$

in which $E_c(0)$ (respectively $G_c(0)$) is the Young's (respectively shear) modulus of delayed elasticity and v_c, the Poisson ratio of the viscoelastic core, is assumed to be frequency independent. The unknowns $\{W(x), B(x), \omega_0\}$ are real quantities. Eigenvalue problem (17), (18) can be solved numerically by finite element method using a specific finite element [11,12,14,16,18,20,21] or analytically by various procedures. An exact analytical solution in the simply supported case is presented in the Appendix A and it will be used in the non-linear analysis.

For the study of non-linear harmonic vibrations, the response is assumed to be harmonic and proportional to the linear vibration mode $\{W(x), B(x)\}$. Based on the one-mode Galerkin's procedure, the deflection and rotation functions are sought in the following forms:

$$w(x,t) = AW(x)e^{i\omega t} + CC,$$
$$\beta(x,t) = AB(x)e^{i\omega t} + CC, \tag{19}$$

where A is a complex unknown amplitude. This approximation assumes that the frequency ω is near the real frequency ω_0 associated to the undamped beam. This allows the consideration that the viscoelastic shear modulus at ω does not differ significantly from its values for ω_0. This approximation will also be done in the following for $E_c(\omega)$ and $E_c(2\omega)$.

3.2. Solution of the axial problem

As usual with von Karman geometrical models, axial problem (16a) is linear with respect to the axial unknowns $u(x,t)$ and $N(x,t)$. From approximation (19), the non-linear term in the strain induces harmonics 0 and 2ω. Thus the axial response can be split in the same manner:

$$u(x,t) = |A|^2 u_0(x) + \{A^2 u_{2\omega}(x)e^{2i\omega t} + CC\},$$
$$N(x,t) = |A|^2 N_0(x) + \{A^2 N_{2\omega}(x)e^{2i\omega t} + CC\}, \tag{20}$$

where $u_0(x)$ and $N_0(x)$ are time independent and $u_{2\omega}(x)$ and $N_{2\omega}(x)$ are the amplitudes of the 2ω-harmonic response. Clearly, they are characterized by the following linear equations:

$$\int_0^L u_0' \delta u' \, dx = -\int_0^L (W')^2 \delta u' \, dx,$$
$$N_0(x) = (2E_f S_f + E_c(0)S_c)(u_0' + (W')^2), \tag{21}$$

$$\int_0^L u_{2\omega}' \delta u' \, dx = -\int_0^L \tfrac{1}{2}(W')^2 \delta u' \, dx,$$
$$N_{2\omega}(x) = (2E_f S_f + E_c(2\omega)S_c)\left(u_{2\omega}' + \tfrac{1}{2}(W')^2\right). \tag{22}$$

It is not difficult to solve these equations. For immovable ends ($u(0) = u(L) = 0$), the solutions of Eqs. (21), (22) can be written in the following form:

$$u_0(x) = \frac{x}{L}\int_0^L (W'(s))^2 \, ds - \int_0^x (W'(s))^2 \, ds,$$
$$u_{2\omega}(x) = \frac{u_0(x)}{2},$$
$$N_0(x) = \frac{(2E_f S_f + E_c(0)S_c)}{L}\int_0^L (W'(s))^2 \, ds,$$
$$N_{2\omega}(x) = \frac{(2E_f S_f + E_c(2\omega)S_c)}{2L}\int_0^L (W'(s))^2 \, ds. \tag{23}$$

As for the case of the cantilever sandwich beam ($N(L) = 0$ or $N(0) = 0$), one finds easily that the normal forces are zero: $N_0(x) = N_{2\omega}(x) = 0$.

Because of the specific geometry of a straight beam, the displacement amplitudes $u_0(x)$ and $u_{2\omega}(x)$ are real, but not the corresponding stress $N_{2\omega}(x)$, which is due to the influence of the complex modulus $E_c(2\omega)$. Two different moduli appear in Eqs. (21), (22), this follows directly from representation (9) of the viscoelastic law. In accordance with the previous approximations, $E_c(2\omega)$ will be replaced by $E_c(2\omega_0)$.

3.3. The non-linear frequency–amplitude relationship

The aim of this paper is to get an approximate solution of the non-linear bending equation (16b). This is done by coupling the one-mode Galerkin approximation with the harmonic balance method. Inserting Eqs. (19)–(22) into the variational bending equation (16b) and assuming that $\delta W = W(x)e^{-i\omega t}$, $\delta\beta = B(x)e^{-i\omega t}$ an amplitude equation following various harmonics is obtained. Based on harmonic balance method, the following frequency–amplitude equation is obtained:

$$-\omega^2 MA + KA + K_{nl}\bar{A}A^2 = Q. \tag{24}$$

The four constants appearing in the amplitude equation (24) are functions of the real linear mode $W(x), B(x)$, of the solution (23) of the axial problem and of the data. They are given by

$$\begin{aligned}
M &= (2\rho_f S_f + \rho_c S_c)\int_0^L W^2(x)\,\mathrm{d}x,\\
K &= \int_0^L \left\{ E_f\left(2I_f + \frac{S_f h_f^2}{2}\right) W''^2 - E_f S_f h_f h_c B' W'' \right.\\
&\quad \left. + \left(E_c(\omega_0) I_c + \frac{E_f S_f h_c^2}{2}\right) B'^2 + G_c(\omega_0) S_c (W' + B)^2 \right\} \mathrm{d}x,\\
K_{nl} &= \int_0^L (N_0 + N_{2\omega})(W')^2\,\mathrm{d}x,\\
Q &= \int_0^L F(x) W(x)\,\mathrm{d}x.
\end{aligned} \tag{25}$$

The real number M corresponds to a modal mass and Q to a modal force which is also real because of the assumption $F(x,t) = F(x)\cos\omega t$. K and K_{nl} represent the linear and non-linear modal stiffness coefficients, respectively. These latter coefficients are complex because of the viscoelastic moduli. The analytical computation of these coefficients is straightforward in the simply supported case and reported in Appendix B. Of course, such computations could also be achieved by the finite element method for more complicated cases. This will be presented later for the analysis of viscoelastic sandwich plates and shells.

4. Free vibration analysis from the amplitude equation

In the previous section, it has been established that, close to resonance, the non-linear vibration of a viscous sandwich beam is approximately governed by the complex amplitude equation (24). In this section, the free vibration problem is discussed from the latter amplitude equation for $Q = 0$. This will explain the physical meaning of the real and imaginary parts of the modal stiffness constants:

$$K = K^R + \mathrm{i}K^I, \quad K_{nl} = K_{nl}^R + \mathrm{i}K_{nl}^I. \tag{26}$$

4.1. Linear analysis and loss factor

The linearized version of the amplitude equation (24) leads to an approximate value of the complex eigenfrequency that can written in the following classical form:

$$\omega^2 = K/M = \Omega_l^2\,[1 + \mathrm{i}\eta_l], \tag{27}$$

where η_l is the loss factor and Ω_l is the linear frequency. So, the viscoelastic frequency and the loss factor of the viscoelastic structure are related to the real and imaginary parts of the linear stiffness by the following relationship:

$$\Omega_l^2 = K^R/M, \quad \eta_l = K^I/K^R. \tag{28}$$

The detailed expressions of these quantities are presented in Appendix B. They coincide with those predicted by the modal stain energy method [10], when it is based on the undamped mode.

4.2. Non-linear response, backbone curve, non-linear loss factor

For free vibration analysis, the amplitude equation (24) leads to a relationship between the complex frequency ω^2 and the real amplitude $a = |A|$. The non-linear frequency and the non-linear loss factor are defined in the same way as the corresponding linear quantities and given by

$$\begin{aligned}
&\omega^2 = \Omega_{nl}^2(1 + \mathrm{i}\eta_{nl}),\\
&\Omega_{nl}^2 = \Omega_l^2\left(1 + C^R\left(\frac{a}{h}\right)^2\right), \quad C^R = \frac{K_{nl}^R}{M\Omega_l^2}h^2,\\
&\eta_{nl} = \eta_l\frac{1 + C^I(a/h)^2}{1 + C^R(a/h)^2}, \quad C^I = \frac{K_{nl}^I}{M\eta_l\Omega_l^2}h^2,
\end{aligned} \tag{29}$$

where h is the thickness of the sandwich beam. For simplicity, non-dimensional constants C^R and C^I have been introduced instead of the real and imaginary parts of the non-linear stiffness.

Hence the amplitude equation (24) permits one to define a frequency and a loss factor, that depend non-linearly on the vibration amplitude. With the second relation (29b), the classical parabolic approximation of the backbone curve (2) is recovered. The non-linear frequency may be greater or lower than the linear one, according to the sign of the constant C^R.

Furthermore, a new amplitude–loss factor relation has been deduced which is given by the rational fraction (29c). The loss factor is a decreasing function of the amplitude if C^R is greater than C^I and increasing in the opposite case.

4.3. Numerical applications

To our knowledge, the concept of an amplitude-dependent damping has not yet been presented, except in a recent paper by Ganapathi et al. [34]. For the sake of validation and of comparison, the case of the simply supported beam is considered as in Ref. [34], the geometrical and material data being given in Table 1. The axial displacement is assumed to be zero at the ends of the beam. The damped core modulus is given by the following simple expression:

$$E_c = E_0(1 + \mathrm{i}\eta_c), \tag{30}$$

where the real modulus E_0 and the material loss factor η_c do not depend on the frequency.

With the chosen boundary conditions, there are simple exact solutions of the linear eigenproblem (17)–(18) as well as closed form expressions of the stiffness constants K and K_{nl}, of the frequency and of the loss factor. They are presented in Appendices A and B.

Three values of the thickness ratio h_f/h_c $(1/7, 1, 7)$ are considered and three values of the shear modulus G_0 (2.5, 25, 2500 MPa). Note that the kinematical assumptions of Section 2.1 are accurate only for the smallest thickness ratio (1/7) and for the two smallest values of the shear modulus (2.5, 25), see Ref. [21]. In Table 2, the numerical values of the linear quantities η_l, Ω_l and of the non-dimensional constants C^R and C^I are presented. As the constant C^R is positive, the frequency increases with the vibration amplitude. This increase is significant because ω^2 at least doubles (i.e., $C^R \geqslant 4$) for a maximal deflection equal to the thickness (i.e. $a/h = 1/2$ with definition (19) of the amplitude). One can note that the constant C^R is much larger than C^I in the soft core cases ($G_0 = 2.5$ or 25 MPa). Indeed, it is obvious from formula (B.9) that the imaginary part of K_{nl} can be neglected in these cases. Hence, for soft cores the non-linear stiffness K_{nl} can be assumed real, it depends on the face properties and its approximated value is given by the following formula:

$$K_{nl} \approx 3E_f S_f \frac{\pi^4}{4L^3}. \tag{31}$$

Because $C^I \approx 0$ in the case of soft cores, the loss factor is a decreasing function of the vibration amplitude, see formula (29).

The variations of the loss factor with the vibration amplitudes are presented in Figs. 2 and 3, for the chosen values of the thickness ratio and of the core modulus. Results obtained by Ganapathi

Table 1
Material and structural data. Free vibration test

Elastic layers
Young's modulus: 45.54 GPa
The Poisson ratio: 0.33
Mass density: 2040 kg/m^3
Viscoelastic layer
Elasticity modulus is varied as 7.2, 72.5, 7250 MPa
The Poisson ratio: 0.45
Mass density: 1200 kg/m^3
Material loss factor: 0.5
Aspect ratios
Thickness ratio is varied as 1/7, 1, 7
Aspect ratio (L/h) 70

Table 2
Analytical values of the linear frequency Ω_l, of the linear loss factor η_l and the non-dimensional constants $C^R$and C^I for various core properties

h_f/h_c	G_c (MPa)	Ω_l (s^{-1})	η_l	C^R	C^I
1/7	2.5	1.51	3.02×10^{-1}	9.7	2.97×10^{-3}
	25	2.25	6.85×10^{-2}	4.39	5.92×10^{-2}
	2500	2.58	6.27×10^{-2}	4.55	4.91
1	2.5	1.79	2.67×10^{-1}	16.38	8.12×10^{-4}
	25	2.66	7.68×10^{-2}	7.40	1.27×10^{-2}
	2500	2.90	3.97×10^{-3}	6.54	20.72
7	2.5	2.35	1.30×10^{-1}	11.82	1.72×10^{-4}
	25	2.72	2.12×10^{-2}	8.78	7.83×10^{-3}
	2500	2.78	2.50×10^{-4}	8.47	63.58

Simply supported sandwich beam. Fixed axial displacement. Geometrical and material data of Table 1. Study of the first vibration mode.

et al. [34] using finite elements and an iterative QR method are reported for comparison. Clearly, there is a very good agreement between the two approaches, except in the stiff core cases. As explained previously, the loss factor decreases with the amplitude in soft core cases. Indeed, in these cases the non-linear effects are mainly due to the axial stress in the elastic layer. This implies a real value of the non-linear stiffness K_{nl}. Hence, this non-linearity is not affected by the damping, and the dependence of the loss factor with respect to the amplitude follows only from the amplitude dependence of the frequency (see Eq. (29) for $C^I = 0$), which leads to a decrease of η_{nl}. For stiff cores, the core axial stiffness cannot be neglected and the imaginary part C^I becomes significant: this implies the increase of the loss factor observed in Fig. 3.

The agreement between the present results and those of Ref. [34] can be considered as a validation of the two approaches. Although they are quite different and there exist restrictive assumptions in each one.

4.4. Sandwich with soft and thin core

Many papers and many applications are devoted to sandwich beams with a soft and thin core ($h_c \ll h_f$, $E_0 \ll E_f$). A discussion of this case is now presented, to specify the influence of the classical adimensional shear parameter g. Further, the complex modulus is not frequency dependent, as in Eq. (30). In classical linear damping analyses [2,5,8,11], the results depend mainly on two adimensional parameters, that are called the shear parameter and the geometrical parameter, respectively

$$g = \frac{2G_0 S_c}{E_f S_f}\left(\frac{L}{h_c}\right)^2,$$

$$Y = \left(\frac{h_f + h_c}{2}\right)^2 \frac{S_f}{I_f}. \tag{32}$$

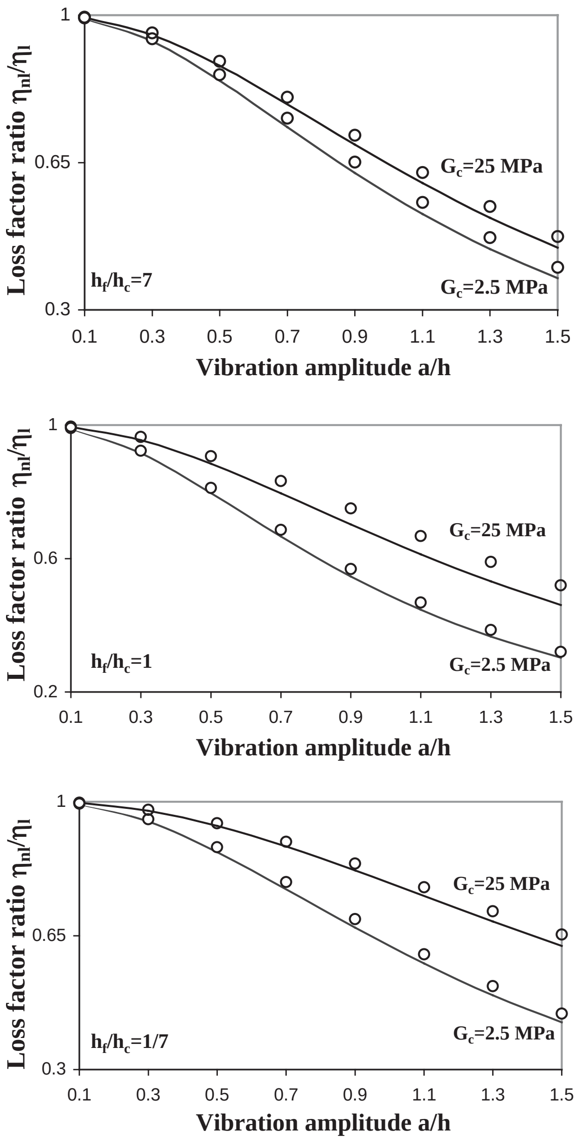

Fig. 2. Loss factor ratio (η_{nl}/η_l) as a function of the vibration amplitude (a/h) for different core properties. Data of Table 1. Soft core cases. Simply supported beam. Fixed axial displacement. Study of the first vibration mode. (○) Ganapathi et al. results [34]; (continuous line) present analytical results.

With the previous assumptions, the frequency Ω_l, the loss factor η_l and the constant C^R can be written as functions of the parameters g, Y and of the Euler–beam frequency Ω_E (Appendix C). By considering the case of a rectangular cross-section, the value of the geometrical parameter Y is about 3. So, the frequency Ω_l, the loss factor η_l and the constant C^R depend only on the shear parameter g. These quantities can be written

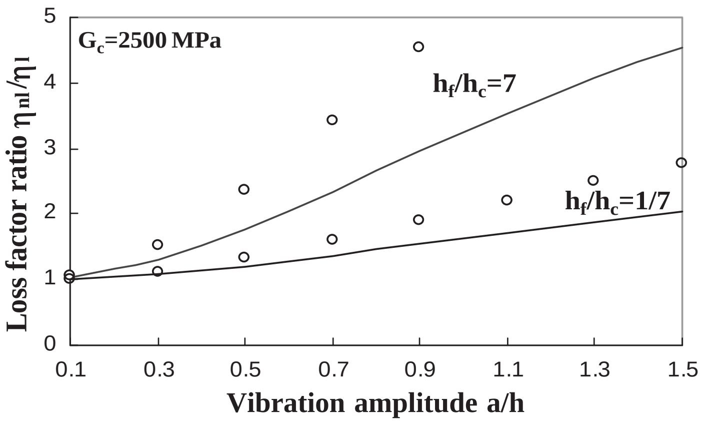

Fig. 3. Loss factor ratio (η_{nl}/η_l) as a function of the vibration amplitude (a/h) for different core properties. Data of Table 1. Stiff core case: $G_0 = 2500$ MPa. Simply supported beam. Fixed axial displacement. Study of the first vibration mode. (○) Ganapathi et al. results [34]; (continuous line) present analytical results.

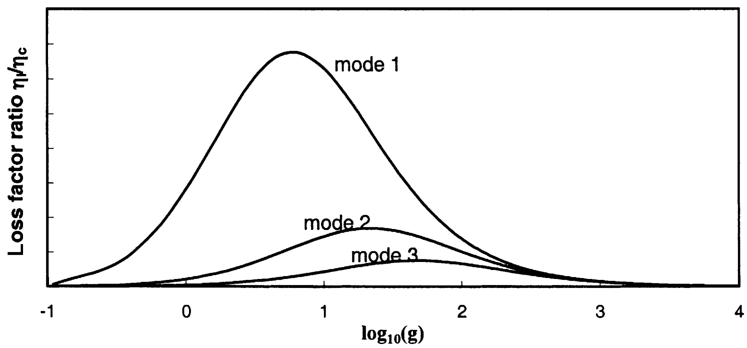

Fig. 4. Loss factor ratio η_l/η_c as a function of the shear parameter g. Thin and soft core approximation. Simply supported beam with rectangular section. Fixed axial displacement. Study of the three first vibration modes.

as follows:

$$\Omega_l = \Omega_E \sqrt{1 + \frac{3g}{((n\pi)^2 + g)}},$$
$$\eta_l = \eta_c \frac{3(n\pi)^2 g}{((n\pi)^4 + 5(n\pi)^2 g + 4g^2)},$$
$$C^R = \frac{36((n\pi)^2 + g)}{(n\pi)^2 + 4g}. \quad (33)$$

Figs. 4–6 present the variation of the loss factor ratio η_l/η_c, of the constant C^R and of the frequency ratio Ω_l/Ω_E, with respect to the shear parameter g. A good agreement is obtained between the results of the ratios $\Omega_l/\Omega_E, \eta_l/\eta_c$ and those obtained by Rao [5]. One can note that the non-linear constant C^R decreases with the shear parameter g and varies from 9 to 36.

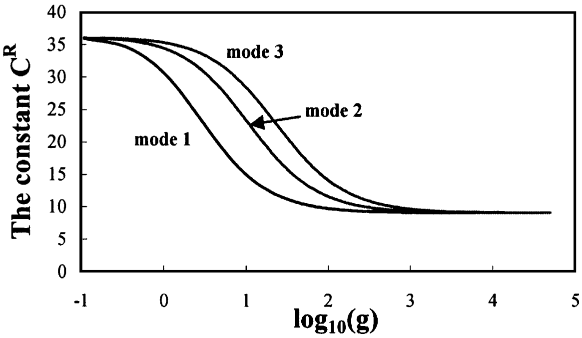

Fig. 5. Constant C^R as a function of the shear parameter g. Thin and soft core approximation. Simply supported beam with rectangular section. Fixed axial displacement. Study of the three first vibration modes.

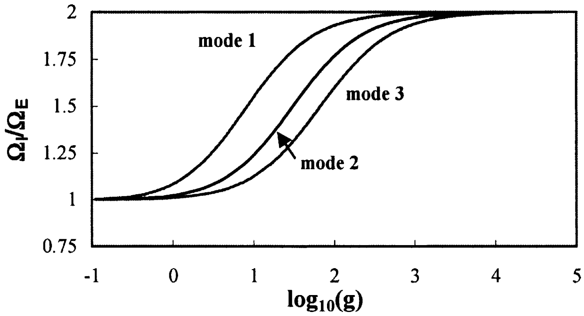

Fig. 6. Frequency ratio Ω_l/Ω_E as a function of the shear parameter g. Thin and soft core approximation. Simply supported beam with rectangular section. Fixed axial displacement. Study of the three first vibration modes.

5. Forced vibration analysis from frequency–amplitude equation

In this section, forced vibrations are studied from the reduced frequency–amplitude equation (24). The first aim is to check that the approximate model keeps the main features of classical non-linear resonance analyses [22]. Next, two examples will be presented, especially to illustrate the influence of the axial boundary conditions on the response curve.

5.1. Solving the frequency–amplitude equation

The non-linear equation (24) can be simply rewritten and solved in the general case where Q is not equal to zero. This Q is assumed to be real, which corresponds to a choice of the forcing phase. The complex numbers A, K and K_{nl} and Eq. (24) are rewritten as

$$\begin{aligned} & -\omega^2 rM + r|K|\mathrm{e}^{\mathrm{i}\varphi} + r^3|K_{nl}|\mathrm{e}^{\mathrm{i}\psi} = Q\mathrm{e}^{-\mathrm{i}\theta}, \\ & A = r\mathrm{e}^{\mathrm{i}\theta}, \\ & K = |K|\mathrm{e}^{\mathrm{i}\varphi}, \\ & K_{nl} = |K_{nl}|\mathrm{e}^{\mathrm{i}\psi}. \end{aligned} \tag{34}$$

Next, the complex equation (34) is split into real and imaginary parts

$$- \omega^2 rM + r|K|\cos\varphi + r^3|K_{nl}|\cos\psi = Q\cos\theta,$$
$$r|K|\sin\varphi + r^3|K_{nl}|\sin\psi = -Q\sin\theta. \quad (35a, b)$$

Combining the last equations (i.e. $(35a)^2 + (35b)^2 = Q^2$) and after some manipulation, one obtains the following frequency–amplitude relation:

$$\omega^4 M^2 - 2\alpha\omega^2 M + \beta = 0, \quad (36)$$

where

$$\alpha = |K|\cos\varphi + r^2|K_{nl}|\cos\psi,$$
$$\beta = |K|^2 + r^4|K_{nl}|^2 + 2r^2|KK_{nl}|\cos(\varphi - \psi) - \frac{Q^2}{r^2}. \quad (37)$$

By this way, the frequency–amplitude curve is obtained in the form $\omega(r)$ from Eqs. (37) and (38).

$$\omega(r) = \sqrt{\alpha \pm \sqrt{\alpha^2 - \beta}}. \quad (38)$$

5.2. Numerical applications

Two numerical illustrations will be presented shortly. They mainly differ by the axial boundary condition that strongly influences the non-linearity of the response.

5.2.1. Cantilever sandwich beam

This example concerns a cantilever sandwich beam, which has been studied quite extensively from theoretical and experimental points of view [5,10,19]. The beam dimensions and material properties are recalled in Table 3: this case concerns sandwich beams with a thin and soft core. The core viscoelastic law is accounted for by a complex constant modulus, as in Eq. (30). The boundary conditions are: $u(0) = 0, w(0) = 0, \beta(0) = 0$ and $N(L) = 0$, $M_w(L) = 0$, $M_\beta(L) = 0$. The key point is the axial boundary condition $N(L) = 0$, as explained in Section 3.2, and this implies that the normal forces $N_0(x)$, $N_{2\omega}(x)$ are zero. Therefore, the non-linear stiffness K_{nl} given by Eq. (25) is also zero. So, because of the axial boundary conditions $N(L) = 0$, the non-linear frequency–amplitude relationship (24) is reduced to the classical linear resonance equation:

$$-\omega^2 MA + KA = Q. \quad (39)$$

The vibrations of this viscoelastic cantilever beam have been studied by Mead [2] for $Q \neq 0$ and by Rao for $Q = 0$ [5]. In Ref. [5], the complex eigenvalues problem has been solved analytically for various boundary conditions. This leads to a better approximation of the loss factor η_l and of the eigenfrequency Ω_l than formula (28), which is based on a modal approximation with the undamped mode. That is why, instead of Eq. (28), Rao's formulae have being used to define η_l and Ω_l and next (27) (39) to define the resonance curve $r(\omega)$ for the first eigenmode which is reported in Fig. 7.

Table 3
Cantilever beam. Geometrical and material data

Elastic layers
Young's modulus: 6.9×10^{10} Pa The Poisson ratio: 0.3 Mass density: 2766 kg/m^3 Thickness: 1.524 mm
Viscoelastic layer
Delayed elasticity modulus 1794×10^3 Pa The Poisson ratio: 0.3 Mass density: 968.1 kg/m^3 Thickness: 0.127 mm
Whole beam
Length 177.8 mm Width 12.7 mm

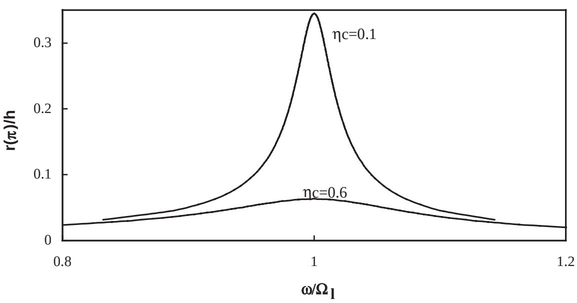

Fig. 7. Cantilever beam. Data of Table 3. $Q = 10$. Resonance curves for $\eta_c = 0.1$ and $\eta_c = 0.6$ Fig. 8. Supported beam. Data of Table 4. Frequency-dependent material. Temperature: 60°C, linear and non-linear frequency–amplitude curves.

5.2.2. Simply supported sandwich beam

The present reduction analysis has been claimed to apply to any viscoelastic model. According to a generalized Maxwell model, a frequency-dependent viscoelastic modulus is considered in this example. All coefficients of this model have been obtained from experimental tests on a pure polymer, as described in Ref. [36]. These material properties are also temperature dependent and this will also permit us to discuss the influence of the temperature on the response curve of a steel/polymer/steel sandwich. The beam dimensions and the main material properties are presented in Table 4: this is also a case of thin and soft core.

This sandwich beam is simply supported and the axial displacement is fixed at the ends of the beam ($u(0) = u(L) = 0$). From formula (A.5), the first eigenfrequency ω_0 ($n = 1$) is equal to $2\pi * 74.23$. Here, the analysis is always limited to this first eigenmode. The obtained values for $\omega^2 = K/M$ (i.e., in the linear range) are presented in Table 5 for various temperatures and

Table 4
Simply supported beam

Elastic layers
Young's modulus: 2.1×10^{11} Pa Poisson's ratio: 0.3 Mass density: 7800 kg/m^3 Thickness: 0.6 mm
Viscoelastic layer
Delayed elasticity modulus 27.216×10^5 Pa Poisson ratio: 0.44 Mass density: 1200 kg/m^3 Thickness: 0.045 mm
Whole beam
Length 178 mm Width 10 mm

Geometrical and material data. Since the core modulus $E_c(\omega)$ involves too many coefficients, the detail of this function are not presented.

Table 5
Simply supported beam

	Present method		Rao's formula [5]	
Temperature	Real part	Imaginary part	Real part	Imaginary part
30	2.78×10^5	2.78×10^4	2.87×10^5	2.40×10^4
50	2.55×10^5	1.31×10^4	2.57×10^5	1.28×10^4
80	2.26×10^5	1.29×10^4	2.27×10^5	1.27×10^4

Data of Table 4. K/M values for the first mode.

compared with those obtained from Rao's formula [5]. The two sets of results are very close, except for the imaginary part (damping) at 30°C, where there is a discrepancy of 15%. This comparison yields a measure of the error due to the real mode approximation (19). These results are consistent with previous ones comparing the modal strain energy method with a real and a complex mode; see for instance Ref. [10]. One can refer also to Refs. [19,21] for a few comparisons of such models with experiments.

Considering a case with a soft core, the non-linear effects are due to the axial stresses in the elastic faces. Thus, the non-linear stiffness K_{nl} can be assumed to be real and given by Eq. (31). The validity of the latter approximation is checked in Table 6 for a wide range of temperatures. In Fig. 8, the linear and non-linear frequency–amplitude curves $r(\omega)$ at 60°C are presented for various amplitudes of excitation ($Q = 0, 5, 10, 15$). The non-linear effect appears clearly and bends to the right the resonance peaks. Such curves are well known in the literature concerning non-linear vibrations [22]. Thus the present theory is consistent with these classical non-linear

Table 6
Simply supported beam

Temperature (°)	Real part of K_{nl}	Imaginary part of K_{nl}
10	1.63×10^{10}	8.70×10^{4}
20	1.63×10^{10}	2.10×10^{4}
60	1.63×10^{10}	1.03×10^{3}
70	1.63×10^{10}	1.00×10^{3}
80	1.63×10^{10}	7.94×10^{2}

Data of Table 4. Values of K_{nl} for the first mode at various temperatures.

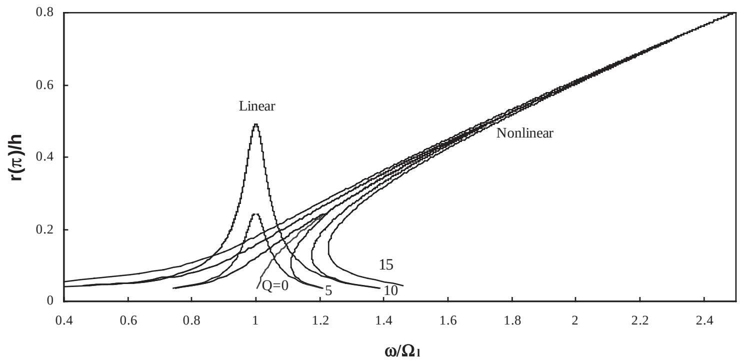

Fig. 8. Simply supported beam. Data of Table 4. Frequency-dependent material. Temperature: 60°C linear and non-linear frequency–amplitude curves.

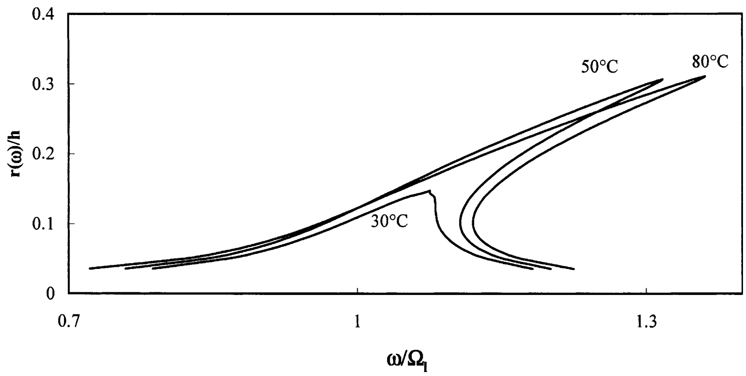

Fig. 9. Simply supported beam. Data of Table 4. Frequency-dependent material. Non-linear frequency–amplitude curves at various temperatures with $Q = 5$.

resonance analyses. The influence of the temperature on the response curves is shown in Fig. 9. The maximal amplitude is smaller at 30°C, because the material damping is greater for this temperature: indeed the core damping η_c, estimated at the beam frequency, is respectively 0.53, 0.21, and 0.19 for 30°C, 50°C and 80°C.

The backbone curve associated to ($Q = 0$), defines the relationship between the natural frequency and the amplitude of free undamped vibrations. The stiffness increases with the deflection and therefore the natural frequency increases as the amplitude increases. This characterizes a system with hardening behaviour. Classically such response curves imply the existence of a downward and of an upward jump, when the excitation frequency varies.

6. Conclusion

An analytical study of free and forced vibrations of elastic/viscoelastic/elastic sandwich non-linear beams has been presented. The beam model and the approximation process have been designed to provide the simplest but most significant model. The sandwich beam modelling is accurate for soft and rather thin cores. The approximation process is based on the harmonic balance method coupled with a one-mode Galerkin approximation. So, it is able to account for near resonance responses with moderately large amplitude. The modelling holds well, whatever is the viscoelastic constitutive law.

In this way, a non-linear frequency–amplitude relationship has been deduced, that involves only a few coefficients. In the free vibration case, only four real numbers are associated with each vibration mode. Two of them characterize the linear behaviour; this permits one to recover the linear eigenfrequency and the loss factor. Two other numbers, denoted by C^R and C^I, account for the non-linear effects. The first one C^R coincides with the classical backbone constant that characterizes the amplitude dependence of the frequency. Similarly, this approach permits a definition of a loss factor that depends also on the amplitude. To the knowledge of the authors, the concept of an amplitude-dependent loss factor has been discussed only in a recent paper [34] and the corresponding formula (29) is new. For a harmonic forcing, the theory permits one to recover classical non-linear resonance curves.

When applied to a sandwich beam with a thin and soft core, this yields a positive value of the constant C^R, the other one C^I being about zero. In such a case, the frequency increases with the amplitude and the loss factor decreases. With a stiffer core, C^I can be significant and the loss factor can increase with the amplitude. As usual for non-linear resonance or post buckling, the non-linear behaviour depends on axial boundary conditions: for instance if an end is stress free, the non-linear effects disappear.

The amplitude–frequency relationship obtained is similar to a bifurcation equation. It is likely that this relation is generic, i.e., it could be valid for the non-linear resonance of any structure with moderately large damping. By comparison with post buckling, a hardening behaviour can be expected for plates ($C^R > 0$) and softening for most of curved shells ($C^R < 0$).

Acknowledgements

This paper was partially written while Azrar enjoyed the hospitality of "LPMM, University of Metz, France" and of the "Aerospace Engineering Department, Old Dominion University, Norfolk, VA, USA". Azrar also wishes to acknowledge the assistance of a fellowship grant from Fulbright.

Appendix A. Analytical solution of the eigenvalue problem (17) for simply supported beam

From Eq. (17) and using two integrations by parts, one finds the following equations:

$$M'_\beta - T = 0, \tag{A.1}$$

$$(M_w)'' - T' = (2\rho_f S_f + \rho_c S_c)\omega_0^2 W. \tag{A.2}$$

Exact solutions of Eqs. (A.1), (A.2) and of the constitutive equations (18), in the case of a simply supported beam are sought, the boundary conditions being expressed by $W = M_\beta = M_w = 0$. In this case, Eqs. (A.1), (A.2), (18) have a family of exact solutions in the form

$$W(x) = \sin(kx), \quad B(x) = b\cos(kx), \quad k = \frac{n\pi}{L}, \tag{A.3}$$

where the integer n is the mode number. Inserting Eqs. (18.a), (18.c) and (A.3) into Eq. (A.1), one gets the number b:

$$b = \frac{(h_c h_f E_f S_f k^3 - 2G_c(0)S_c k)}{[(E_f S_f h_c^2 + 2E_c(0)I_c)k^2 + 2G_c(0)S_c]}. \tag{A.4}$$

Inserting Eqs. (18), (A.3) and (A.4) into (17) or (A.2), one gets the expression of the linear frequency

$$\omega_0 = \sqrt{\frac{(2E_c(0)I_c + E_f S_f h_c^2)k^2 b^2 - 2E_f S_f h_c h_f b k^3 + E_f(4I_f + S_f h_f^2)k^4 + 2G_c(0)S_c(k+b)^2}{2(2\rho_f S_f + \rho_c S_c)}}. \tag{A.5}$$

Appendix B. Analytical expression of K_{nl} for sandwich beam with immovable ends

The real and imaginary parts of the Young's and of the shear modulus of the viscoelastic core are introduced:

$$\begin{aligned} E_c(\omega_0) &= E_c^R(\omega_0) + \mathrm{i}E_c^I(\omega_0), \\ G_c(\omega_0) &= G_c^R(\omega_0) + \mathrm{i}G_c^I(\omega_0). \end{aligned} \tag{B.1}$$

Using these decompositions and Eqs. (25) and (27) one obtains Ω_l and η_l as functions of the linear mode

$$\begin{aligned} \Omega_l^2 = \frac{1}{M}\int_0^L \Bigg\{ & E_f\left(2I_f + \frac{S_f h_f^2}{2}\right)W''^2 - E_f S_f h_f h_c B' W'' \\ & + \left(E_c^R(\omega_0)I_c + \frac{E_f S_f h_c^2}{2}\right)B'^2 + G_c^R(\omega_0)S_c(W'+B)^2 \Bigg\}, \end{aligned} \tag{B.2}$$

$$\eta_l = \frac{1}{M\Omega_l^2}\int_0^L \left\{ E_c^I(\omega_0)I_c B'^2 + G_c^I(\omega_0)S_c(W'+B)^2 \right\} \mathrm{d}x. \tag{B.3}$$

The axial problems (21), (22) can be solved as follows:

$$u_0' + W'^2 = \alpha_1,$$
$$u_{2\omega}' + \tfrac{1}{2} W'^2 = \alpha_2, \quad \text{(B.4)}$$

where α_1 and α_2 are constants. The integration of the latter equations leads to

$$L\alpha_1 = \int_0^L u_0' \, \mathrm{d}x + \int_0^L W'^2 \, \mathrm{d}x,$$
$$L\alpha_2 = \int_0^L u_{2\omega}' \, \mathrm{d}x + \frac{1}{2}\int_0^L W'^2 \, \mathrm{d}x. \quad \text{(B.5)}$$

For immovable ends, $u_0(0) = u_0(L) = u_{2\omega}(0) = u_{2\omega}(L) = 0$, these constants are

$$\alpha_2 = \frac{1}{2}\alpha_1 = \frac{1}{2L}\int_0^L W'^2 \, \mathrm{d}x. \quad \text{(B.6)}$$

The coefficient K_{nl} defined in Eq. (25) can be rewritten as a function of the modal deflection

$$K_{nl} = (2E_f S_f + E_c(0)S_c)\alpha_1 \int_0^L W'^2 \, \mathrm{d}x + (2E_f S_f + E_c(2\omega_0)S_c)\alpha_2 \int_0^L W'^2 \, \mathrm{d}x. \quad \text{(B.7)}$$

Thus K_{nl} is in form

$$K_{nl} = \left[(2E_f S_f + E_c(0)S_c) + \tfrac{1}{2}(2E_f S_f + E_c(2\omega_0)S_c)\right] L\alpha_1^2. \quad \text{(B.8)}$$

Using the decomposition (B.1), one finally deduces that K_{nl} is also in form

$$K_{nl} = [6E_f S_f + 2E_c(0)S_c + (E_c^R(2\omega_0) + E_c^I(2\omega_0))S_c]\frac{L}{2}\alpha_1^2. \quad \text{(B.9)}$$

In the simply supported case presented in Appendix A, the value of α_1 is $n^2\pi^2/2L^2$.

Appendix C. Simply supported sandwich beam with soft and thin core

Consider that the core modulus is given by formula (30). Neglecting the term $E_c I_c$ and using Eqs. (A.4), (A.5), (31) and some manipulations, one finds that the constant b, the frequency Ω_l, the loss factor η_l and the constant C^R are functions of the shear parameter g and of the geometric one Y:

$$b = \frac{h_f(n\pi)^2 - gh_c}{h_c((n\pi)^2 + g)},$$
$$\Omega_l = \Omega_E\sqrt{1 + \frac{Yg}{((n\pi)^2 + g)}},$$
$$\eta_l = \eta_c \frac{(n\pi)^2 gY}{((n\pi)^4 + (n\pi)^2(2 + Y)g + (1 + Y)g^2)},$$
$$C^R = Y\left(\frac{h}{h_f + h_c}\right)^2 \frac{3}{1 + (Yg/((n\pi)^2 + g))}, \quad \text{(C.1)}$$

where Ω_E is the Euler beam frequency, given by

$$\Omega_E = \left(\frac{n\pi}{L}\right)^2 \sqrt{\frac{2E_f I_f}{2\rho_f S_f + \rho_c S_c}}. \tag{C.2}$$

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